REFLECTION OF A PLASTIC WAVE FROM AN OBSTACLE

(OTRAZHENIE PLASTICHESKOI VOLNY OT PREGRADY)

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The propagation of a plane wave and its interaction with an obstacle in elasto-plastic media has been investigated by Olisov, Kaliski and Osiecki [1], by Perzyna [2], and by Liakhov and Poliakova [3]. In these investigations a linear or piecewise linear compression diagram was used to approximate to the law of loading. In [1,2], moreover, the hypothesis of constancy of density during unloading was accepted.

The present paper shows that the problem of interaction of a plane wave and an obstacle can be solved comparatively simply if a power-law relation between stress and strain is assumed during loading and if the density within a particle is assumed constant during unloading. Such an approximation constitutes a close description of experimental results for a wide range of stresses.

The paper also studies the parameters of an incident wave in relation to a given external influence, as well as the parameters of a wave reflected from an absolutely rigid or massive movable obstacle.

The results can be applied to the study of wave phenomena in soft soils.

1. Description of the medium. In order to represent the medium we adopt the model proposed for soils by Grigorian [5]. In the case of a plane wave the conditions of coaxiality of the tensors of stress and rate of strain are fulfilled by virtue of symmetry, and to specify the medium there remain the following two conditions:

a) the law of volume compression

$$\sigma = \sigma(\theta) \tag{1.1}$$

b) the plasticity condition [6]

$$|\sigma_x - \sigma_y| = -m\sigma + m' \tag{1.2}$$

Here σ_x, σ_y are components of stress; σ is the mean stress, θ is the volume compression, m and m' are positive constants.

We shall confine our attention in this paper to phenomena which take place under conditions of plastic deformation.

The law of volume compression is assumed to be different for loading and for unloading (Fig. 1):

$$\sigma(\theta) = f_1(\theta) \quad \text{for } d\sigma/dt > 0 \text{ (the line } ABC) \tag{1.3}$$

θ = const, σ is indeterminate for $d\sigma/dt < 0$ (the straight line BD)

If we confine our attention to a plane wave, then we need only consider the law of uniaxial compression, which can be derived from conditions (1.2) and (1.3).

Suppose that the direction of the x-axis coincides with the direction of propagation of the wave. Assuming the stress to be compressive, we have

$$\sigma_y - \sigma_x = -m\sigma + m' \qquad \text{or} \qquad \sigma = \frac{3}{3+2m}\sigma_x + \frac{2m'}{3+2m}$$

It follows from (1.3) (for the case of loading) that

$$\sigma_x = \left(1 + \frac{2m}{3}\right) f_1(\theta) - \frac{2m'}{3} \equiv f(\theta)$$

In the case of uniaxial deformation $\varepsilon_y = \varepsilon_z = 0$, $\theta = \varepsilon_x$. Then, finally

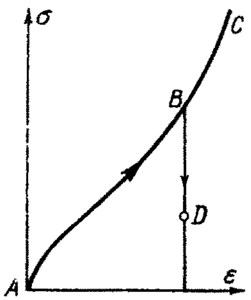
$$\sigma_x = f(\varepsilon_x) \tag{1.4}$$

2. Propagation of a plane wave generated by an external influence. Suppose that a uniform compressive stress is given as a function of time at various points in some plane. As a result waves will start to propagate from these points in both directions. Let us investigate one of these waves, taking the x-axis as the direction of propagation. We shall adopt a system of Lagrangean coordinates (h, t), taking

$$x(h, t) = h + u(h, t), \qquad u(h, 0) = 0 \tag{2.1}$$

Here $x(h,\ t)$ is an Euler coordinate, $u(h,\ t)$ is the displacement of a particle.

The law of uniaxial deformation will be taken in the form (1.4) for loading. For unloading we shall assume that ε_x = const. In accordance with well-known experimental results we assume that with repeated loading the stress within a particle increases without increase in density (i.e. "along the vertical", Fig. 1) until the former maximum stress is reached (point B). Further increase in stress follows the loading branch of the compression diagram (segment BC). From now on, for simplicity, we shall write σ for σ_x and ε for ε_x. Since we propose to consider only the plastic state, we can apply the law as stated, excluding the initial portion of the curve corresponding to the elastic stage.



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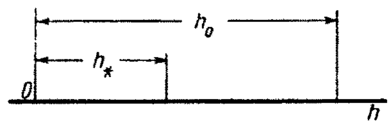

Fig. 2.

Until the application of the external load the medium was in an undisturbed state. We shall assume that the external load is applied instantaneously, and that thereafter its absolute magnitude decreases with time. These assumptions concerning the nature of the external load are essential for the method of solution proposed here. Moreover, they depict a loading which is typical of that applied by explosions. It is natural to expect that a shock-loading will generate a wave with a shock-front. The coordinate of the front (Fig. 2) will be denoted by h_*, and the coordinate of the plane with a given stress, by h_0. Particles with coordinates h, for which $h_* < h < h_0$, are in a state of unloading. In this region we have an equation of motion and an equation of continuity

$$\frac{\partial \sigma}{\partial h} - \rho_0 \frac{\partial v}{\partial t} = 0, \qquad \frac{\partial x}{\partial h} = \frac{\rho_0}{\rho(h)} = 1 + \varepsilon(h) \tag{2.2}$$

Here $v(h,\ t)$ is the particle velocity, ρ_0 and $\rho(h)$ are the values of the initial density and the density after the shock-wave has passed. From the second equation of (2.2) we obtain

$$x(h,t) = \int_{h_0}^{h} \frac{\rho_0 d\eta}{\rho(\eta)} + x_0(t) \tag{2.3}$$

Also

$$v(h, t) = \frac{\partial x}{\partial t} = x_0'(t) \tag{2.4}$$

and consequently the particle velocity is independent of the coordinate and is dependent solely on time. Making use of (2.4), we have from the first equation of (2.2)

$$\sigma(h, t) = \rho_0 x_0''(t)(h - h_0) + C(t) \tag{2.5}$$

In order to specify the wave completely it is necessary to know the functions $x(h, t)$, $v(h, t)$, $\sigma(h, t)$, $\rho(h)$, $\varepsilon(h)$ and $h_*(t)$. From the foregoing we see that it is sufficient to find the functions $x_0(t)$, $C(t)$, $h_*(t)$ and $\rho(h)$ and the remainder also will then be known. To determine these four functions we have the four conditions:

a) the stress on the plane $h = h_0$ is given,

b) the relation between strain and stress on the front (the compression diagram,

c) and d) two mechanical conditions on the shock-front.

We shall show that these conditions are sufficient, and we shall find the solution which they give. It is natural for the external stress at the section $h = h_0$ to be specified as a function of time. Since the distance $h_*(t)$ is also a function of time, and is in addition a monotonic function, we can suppose that the stress $\sigma_0 = \sigma_0(h_*)$ is a known function of h_*.

When the problem has been solved and all the elements of the motion have been found, including the function $h_*(t)$, σ_0 will be known as a function of time, and it will then be clear for which loading law the problem has been solved.

The four conditions can be written in the form

$$\sigma(h_0, t) = \sigma_0(h_*), \qquad \sigma(h_*, t) = f(\varepsilon_*), \qquad \sigma(h_*, t) = \rho_0 \varepsilon_* h_*'^2$$
$$v(h_*, t) = -\varepsilon_* h_*' \tag{2.6}$$

The asterisks indicate quantities referring to the wave-front.

Note that from the third and second equations of (2.6) it follows that

$$f(\varepsilon_*) = \rho_0 \varepsilon_* h_*'^2 \tag{2.7}$$

This relation defines $\varepsilon_* = \varphi(\rho_0 h_*'^2)$ as a single-valued function of

$\rho_0 h_*'^2$ if the function $f(\varepsilon_*)/\varepsilon_*$ is monotonic. Eliminating the functions x_0, C, ρ, we obtain in general a nonlinear first-order equation in the function $h_*'^2$

$$\rho_0 [\varphi(\rho_0 h_*'^2) + 2\rho_0 h_*'^2 \varphi'(\rho_0 h_*'^2)] \frac{dh_*'^2}{dh_*} + \frac{2\rho_0 h'^2_* \varphi(\rho_0 h_*'^2)}{h_* - h_0} + \frac{2\sigma_0(h_*)}{h_* - h_0} = 0 \quad (2.8)$$

Integrating this equation, we obtain

$$\left(\frac{dh_*}{dt}\right)^2 = F(h_*), \quad \text{or} \quad t = \int_{h_0}^{h_*} \frac{dh}{\sqrt{F(h)}}$$

Whenever equation (2.8) can be reduced to a linear equation it can be integrated fairly easily. Let us consider the case when it is possible to do so. Introducing the notation $\rho_0 h_*'^2 = \zeta$, we can reduce equation (2.8) to the form

$$[\varphi(\zeta) + 2\zeta\varphi'(\zeta)] \frac{d\zeta}{dh_*} + \frac{2\zeta\varphi(\zeta)}{h_* - h_0} + \frac{2\sigma_0(h_*)}{h_* - h_0} = 0$$

Suppose that there exists a function $\omega(\zeta)$ such that the preceding equation can be written in the form

$$\frac{d\omega(\zeta)}{dh_*} + \frac{\lambda\omega(\zeta)}{h_* - h_0} + \frac{2\sigma_0(h_*)}{h_* - h_0} = 0$$

where λ is an arbitrary constant.

Then

$$\lambda\omega(\zeta) = 2\zeta\varphi(\zeta), \qquad \omega'(\zeta) = \varphi(\zeta) + 2\zeta\varphi'(\zeta)$$

From this we obtain a differential equation for the function $\varphi(\zeta)$

$$\zeta \frac{d\varphi}{d\zeta} + \frac{2 - \lambda}{2(1 - \lambda)} \varphi = 0$$

The solution to this equation is

$$\varphi(\zeta) = C\zeta^{n_1}, \qquad n_1 = \frac{\lambda - 2}{2(1 - \lambda)}$$

In this way we obtain the following expression for the law of uniaxial compression:

$$\sigma = f(\varepsilon) = C\varepsilon^{-\frac{\lambda}{\lambda - 2}}$$

Thus, by replacing the required function, equation (2.8) can be reduced to a linear first-order equation only when the diagram of uniaxial compression follows a power law. From now on we shall assume that the law of uniaxial compression is described by the function

$$f(\varepsilon) = \sigma^\circ |\varepsilon|^n \qquad (\sigma^\circ, n - \text{const}, \ \sigma^\circ < 0, n > 1) \tag{2.9}$$

The power-law approximation for the compression diagram is valid, in general, for values of ε which are not too small, and it should be remembered that the present theory can only be applied when the event under investigation follows the particular segment of the diagram for which the approximation is made.

In the case of a function given in the form of (2.9), equation (2.8) becomes

$$\frac{dZ}{dh_*} + \frac{2n}{n+1}\frac{Z}{h_* - h_0} + \frac{2n}{n+1}\frac{\sigma_0(h_*)}{\sigma^\circ(h_* - h_0)} = 0, \qquad Z = \left(\frac{\rho_0 h_*'^2}{-\sigma^\circ}\right)^{\frac{n}{n-1}} \tag{2.10}$$

Since it is linear, this equation can be easily integrated. Its general solution is

$$Z = C(h_0 - h_*)^{-\frac{2n}{n+1}} + \frac{2n}{n+1}(h_0 - h_*)^{-\frac{2n}{n+1}} \int_{h_*}^{h_0} \frac{\sigma_0(\xi)}{\sigma^\circ}(h_0 - \xi)^{\frac{n-1}{n+1}} d\xi$$

As $h_* \to h_0$, the first term on the right-hand side becomes infinite, whilst the second term remains finite. The particle velocity is finite (and is finite also at the initial instant) and the constant of integration must therefore be zero. As a result we obtain

$$-h_*' = \left(\frac{-\sigma^\circ}{\rho_0}\right)^{\frac{1}{2}} \left(\frac{2n}{n+1}\right)^{\frac{n-1}{2n}} \Phi(h_*)(h_0 - h_*)^{\frac{1-n}{1+n}} \tag{2.11}$$

where

$$\Phi(h) = \left\{\int_{h}^{h_0} \frac{\sigma_0(\xi)}{\sigma^\circ}(h_0 - \xi)^{\frac{n-1}{n+1}} d\xi\right\}^{\frac{n-1}{2n}}$$

The law governing the motion of the front with time is obtained in the form

$$t = \left(\frac{\rho_0}{-\sigma^\circ}\right)^{\frac{1}{2}} \left(\frac{n+1}{2n}\right)^{\frac{n-1}{2n}} \int_{h_*}^{h_0} (h_0 - \xi)^{\frac{n-1}{n+1}} \Phi^{-1}(\xi)\, d\xi \tag{2.12}$$

From (2.7) the strain on the wave-front is given by

$$-\varepsilon_* = \left(\frac{2n}{n+1}\right)^{\frac{1}{n}} [\Phi(h_*)]^{\frac{2}{n-1}} (h_0 - h_*)^{-\frac{2}{n+1}} \tag{2.13}$$

Knowing the function $\varepsilon(h)$, we can find $\rho(h)$ from the formula

$\rho_0/\rho(h) = 1 + \varepsilon(h)$ and from the Euler coordinate of the particle at an arbitrary instant of time

$$x(h, t) = h + \int_{h_*}^{h} \varepsilon(\eta)\, d\eta \tag{2.14}$$

The velocity $v(h, t)$ of a particle and the stress $\sigma(h, t)$ are given by formulas (2.5) and (2.6). For the stress we obtain

$$\sigma(h, t) = \frac{1}{h_* - h_0}\left[\sigma_0(h_*)(h_* - h) - \rho_0\varepsilon_* h_*'^2 (h - h_0)\right] \tag{2.15}$$

Formulas (2.11) to (2.15) specify completely a wave generated by a stress applied at the section h_0. Formula (2.12) enables us to express the function $\sigma_0(h_*)$ as a function of time. From the results obtained we can draw the following qualitative conclusions:

1. If the external stress acts even for a finite time, the wave-front still continues to propagate an infinite distance and for an infinite time with a velocity which decreases monotonically and tends asymptotically to zero (we can only investigate the case of sufficiently high stresses on the wave-front).

2. The stress $\sigma(h, t)$ varies linearly with the coordinate h between the front and the initial section.

3. The particle velocity within this interval is constant.

3. Reflection from an absolutely fixed obstacle. The wave motion described in Section 2 will from now on be called an "incident wave" and quantities referring to this will be denoted by the suffix 1. The problem now is to determine the new wave motion generated on the collision of an incident wave with an obstacle. This new motion will be called a "reflected wave" and its parameters will be denoted by the suffix 2.

In order to solve problems of nonlinear reflection, it is necessary to make certain assumptions beforehand concerning the properties of the solution. In this case we shall suppose that on reflection the stress increases, and that the reflected wave has a shock-front (Fig. 3). The region between the obstacle and the shock-front will be denoted by the Fig. 2. On the basis of these assumptions a particle in the path of the approaching reflected wave experiences a step-wise increase in stress (in accordance with the compression diagram) followed by an unloading phase. In the region (2), consequently, the following equations hold:

$$\frac{\partial\sigma_2}{\partial h} - \rho_0\frac{\partial v_2}{\partial t} = 0, \qquad \frac{\partial x_2}{\partial h} = \frac{\rho_0}{\rho_2(h)} = 1 + \varepsilon_2(h) \tag{3.1}$$

The conditions on the fixed obstacle, which we shall assume to be at the origin of coordinates, are

$$x_2(0, t) = 0, \quad v_2(0, t) = 0 \qquad (3.2)$$

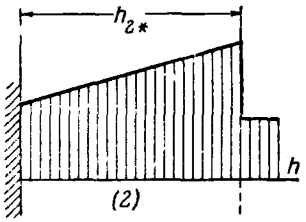

Fig. 3.

From the second equation of (3.1) we obtain

$$x_2(h, t) = \int_0^h \frac{\rho_0 d\eta}{\rho_2(\eta)} + x_{20}(t), \quad v_2(h, t) = x'_{20}(t) \qquad (3.3)$$

and by virtue of (3.2) we have that

$$x_{20}(t) \equiv 0, \quad v_2(h, t) \equiv 0$$

Thus, particles in the region (2) remain fixed; this is in fact a region of no motion in which the particles do not all have the same density, but the stress, which varies with time, is the same for all particles. This conclusion follows from the first equation of (3.1), from which

$$\sigma_2(h, t) = \sigma_2(t) \qquad (3.4)$$

It now remains to determine the functions $\sigma_2(t)$, $\rho_2(h)$, $h_{2*}(t)$; to do so we use two conditions on the shock-front and the compression diagram. The general form of the mechanical conditions on the shock-front is

$$(D - v')\rho' = (D - v_2)\rho_2, \qquad \sigma' - \sigma_2 = (D - v_2)\rho_2(v_2 - v')$$

Here the dashes denote values of the functions ahead of the front of the reflected wave; in what follows methods are given for their determination. The velocity of propagation of the shock-front will be denoted by D. In the present case

$$D = \frac{dx_2(h_{2*})}{dt} = \frac{\rho_0}{\rho_2(h_{2*})} h_2{}'_*, \qquad v_2 = 0$$

Therefore the conditions on the shock-front can be written as

$$v' = \frac{\rho_0}{\rho'}\left(\frac{\rho'}{\rho_{2*}} - 1\right) h'_{2*}, \qquad \sigma' - \sigma_2 = -\rho_0 \frac{\rho_0}{\rho'}\left(\frac{\rho'}{\rho_{2*}} - 1\right) h'^2_{2*} \qquad (3.5)$$

These must be supplemented by the compression law

$$\sigma_2 = \sigma^\circ \left(1 - \frac{\rho_0}{\rho_{2*}}\right)^n \qquad (3.6)$$

We proceed now to a consideration of the events which take place ahead of the reflected wave-front. At the instant of collision between the front of the incident wave and the obstacle there extends a region ahead of the front of the just-formed reflected wave through which the incident wave has passed. The reflected wave propagates through an already disturbed state. The question arises as to whether it will induce ahead of itself some disturbance, or whether it will travel through the same state which was left by the incident wave. The answer to this question is related to the law of deformation of a particle which has undergone an unloading stage. According to our assumption (see the beginning of Section 2), a particle which has undergone unloading in the reflected wave will retain its density during repeated loading until the maximum stress σ_b, to which it was subjected on the front of the incident wave, is reached (the point B in Fig. 2). In this stage the disturbances are propagated with infinite velocity. Therefore, preceding the front of the reflected wave there must be another wave travelling at infinite (in practice very high) velocity.

The region ahead of the front of the reflected wave will be denoted by the Fig. 3, and related quantities by the suffix 3. In fact, this motion should also refer to the reflected wave. Let us determine the motion in the region (3). For this purpose we make a further assumption and equate the stress σ' preceding the shock-front of the reflected wave to the maximum* stress σ_b for a given particle. In the region (3) a particle does not undergo a change of density and therefore $\rho_3 = \rho_1$. Consequently

$$x_3(h, t) = \int_{h_0}^{h} \frac{\rho_0\, d\eta}{\rho_1(\eta)} + x_{30}(t)$$

whereas for the incident wave

$$x_1(h, t) = \int_{h_0}^{h} \frac{\rho_0\, d\eta}{\rho_1(\eta)} + x_{10}(t)$$

Since the displacement is continuous, then in particular

$$x_3(h_{2*}, t) = x_1(h_{2*}, t)$$

This is an identity in t, and it follows that $x_{30}(t) = x_{10}(t)$, i.e.

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* The assumption that $\sigma' = \sigma_b$ is an independent hypothesis in the sense that all the previously imposed conditions can be satisfied for any other choice of σ'. The shock-wave would then be obviously unstable.

that the displacement field in the state (3) coincides with that in the state (1). In other words, the state ahead of the front of the shock-wave is not associated with the appearance of new displacements. Since the medium is incompressible, this does not mean that no new stresses are induced. The equation of motion for state (3) is

$$\frac{\partial \sigma_3}{\partial h} - \rho_0 \frac{\partial v_3}{\partial t} = 0$$

Let $\sigma_3 = \sigma_1 + \sigma_3'$, where σ_1 is the stress in the incident wave, σ_3' is the increase in stress.

Since $v_3 = v_1$, and v_1 and σ_1 satisfy equation (2.2), then obviously

$$\frac{\partial \sigma_3'}{\partial h} = 0, \qquad \sigma_3' = \sigma_3'(t)$$

This leads to the following conclusion: in front of the reflected wave-front a stress is produced which is constant with respect to the coordinate but variable (in general) with time. Its magnitude is determined by the condition that ahead of the front of the reflected wave the stress σ_b is reached.

In order to determine the parameters of the reflected wave we set

$$\rho' = \rho_1(h), \qquad v' = v_1(t), \qquad \sigma' = \sigma_b(h)$$

in equalities (3.5).

As a result we obtain

$$v_1 = \left(\frac{\rho_0}{\rho_2} - \frac{\rho_0}{\rho_1}\right)_{h=h_{2*}} h_{2*}', \qquad \sigma_b - \sigma_2 = -\rho_0 \left(\frac{\rho_0}{\rho_1} - \frac{\rho_0}{\rho_2}\right)_{h=h_{2*}} h_{2*}'^2 \tag{3.7}$$

Eliminating the functions $\rho_2(h)$ and $\sigma_2(t)$ from (3.6) and (3.7), we obtain

$$\sigma^\circ \left[-\varepsilon_1(h_{2*}) - \frac{v_1}{h_{2*}'}\right]^n = \sigma_b + \rho_0 v_1 h_{2*}' \tag{3.8}$$

From the third and fourth equations of (2.6) it follows that

$$v_1 = -\varepsilon_1(h_{1*})\, h_{1*}' = -\frac{\sigma_1(h_{1*})}{\rho_0 h_{1*}'}$$

Equation (3.10) can therefore be reduced to the form

$$\left[1 - \frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})} \frac{h_{1*}'}{h_{2*}'}\right]^n = 1 - \left[\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\right]^n \frac{h_{2*}'}{h_{1*}'} \tag{3.9}$$

Integrating this nonlinear first-order equation with the initial condition $h_{2*}(0) = 0$, we find the function h_{2*}. The stress and density can then be found from the formulas

$$\rho_2(h) = \rho_1(h)\left[1 + \frac{\rho_1(h)}{\rho_0}\frac{v_1(t)}{h'_{2*}(t)}\right]^{-1}, \qquad t = t(h) \equiv t(h_{2*}) \tag{3.10}$$

$$\sigma_2(t) = \sigma_b(h) + \rho_0 v_1(t) h'_{2*}(t), \qquad h = h(t) \equiv h_{2*}(t) \tag{3.11}$$

In order to facilitate the integration of equation (3.9), we can replace it approximately by another equation. From physical considerations we might expect that

$$\frac{\varepsilon_1(h_1)}{\varepsilon_1(h_2)} < 1, \left|\frac{h'_{1*}}{h'_{2*}}\right| < 1$$

and with the aid of these inequalities we can expand the left-hand side of the equation in a series, retaining only the first two terms of the expansion; we then obtain

$$\frac{h'_{2*}}{h'_{1*}} = -\sqrt{n}\left[\frac{\varepsilon_1(h_{2*})}{\varepsilon_1(h_{1*})}\right]^{\frac{n-1}{2}} + \dots$$

or

$$|\varepsilon_1(h_{2*})|^{-\frac{n-1}{2}} dh_{2*} + \sqrt{n}\,|\varepsilon_1(h_{1*})|^{-\frac{n-1}{2}} dh_{1*} = 0 \tag{3.12}$$

Integration of this equation gives the following implicit relation between h_{2*} and h_{1*}:

$$\int_0^{h_{2*}} |\varepsilon_1(h)|^{-\frac{n-1}{2}} dh + \sqrt{n}\int_0^{h_{1*}} |\varepsilon_1(h)|^{-\frac{n-1}{2}} dh = 0 \tag{3.13}$$

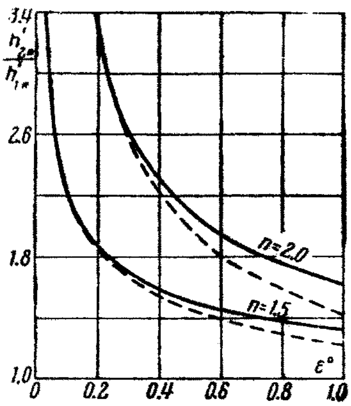

Fig. 4.

An idea of the magnitude of the error introduced by replacing (3.9) by (3.12) is given in Fig. 4, in which the continuous line represents the relation between $h^{\circ} = h'_{2*}/h'_{1*}$ and $\varepsilon_1{}^{\circ} = \varepsilon_1(h_{1*})/\varepsilon_1(h_{2*})$ based on the exact formula (3.9), and the broken line represents the same relation given by formula (3.12).

The calculations were carried out for two values of n. A clear interpretation of the quantities appearing in equation (3.9) (and in the subsequent equations) can readily be obtained by introducing a fictitious incident wave propagating beyond the obstacle.

Figure 5 shows a graph of stress in the reflected wave for a fixed instant of time and a graph of stress in the incident wave for the same instant of time on the assumption that there is no obstacle (Fig. 5, the broken line).

The front of this fictitious wave determines the quantities h_{1*} and $\varepsilon_1(h_{1*})$, whereas the front of the reflected wave denotes a particle with a coordinate h_{2*} and a value of $\varepsilon_1(h_{2*})$ corresponding to this particle.

From the general results obtained the following conclusions can be drawn:

a) the velocity of the front of the reflected wave is always greater than the velocity of the front of the fictitious incident wave;

b) in accordance with the hypotheses made

$$\rho_2 > \rho_1, \quad |\sigma_2| > \sigma_1|$$

c) at the instant of reflection $h_{1*} = h_{2*} = 0$, and therefore for a given instant of time the ratio $h'_{2*}/h'_{1*} = q$ can be found from the equation

$$(1 + q^{-1})^n = 1 + q \tag{3.14}$$

Below are given values of q calculated for various values of n

$q =$	1.0	1.5	2.0	2.5	3.0
$n =$	1.0	1.32	1.62	1.84	2.15

Since at the instant of reflection $\sigma_{1*} = \sigma_b$, we find from (3.11) that

$$\left(\frac{\sigma_2}{\sigma_b}\right)_{t=t_0} = 1 + q \tag{3.15}$$

This ratio can be called the coefficient of reflection, and in the present formulation of the problem depends only on the properties of the medium (the exponent n) and is independent of the intensity of the incident wave. Since $q > 1$, the coefficient of reflection is greater than two;

d) the observer can imagine himself to be attached to a fixed particle and study the variation with time in its state of stress.

The foregoing analysis leads to the result shown in Fig. 6, where time is measured along the axis of abscissae and the stress in the particle along the axis of ordinates.

Until the instant $t = t_p$, the particle is in a state of rest, and at $t = t_p$ it is reached by the front of the incident wave.

The stress increases step-wise to σ_b, and then follows an unloading stage which lasts until the instant $t = t_0$, is the instant of collision of the incident wave with the obstacle.

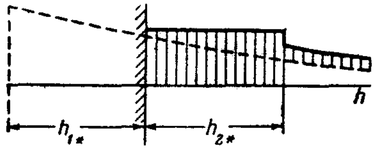

Fig. 5.

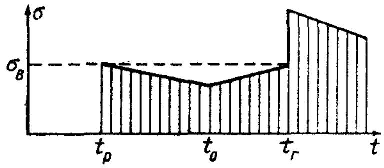

Fig. 6.

At this same instant the "forerunner" of the reflected wave reaches the particle (its velocity of propagation is infinite). Its intensity increases from zero (since at the obstacle $\sigma_b = \sigma_{1*}$).

The stress in the particle continues to increase and by the instant $t = t_r$ of the arrival of the front of the reflected wave reaches once more the value σ_b.

As the front passes, the stress again increases step-wise. Starting from this instant the particle becomes fixed and the stress decreases with time.

4. Particular cases of loading. a) Suppose that the external stress $\sigma_0(h_{1*})$ instantaneously reaches the value σ_0 and thereafter remains constant; in other words, suppose that the application of the given loading follows a step law. In this case the functions defining the incident and reflected waves also have a step-like character. The constant values of these functions can easily be found from the preceding formulas. For the incident wave we find that in this case

$$h_{1*}' = -\left(\frac{-\sigma^\circ}{\rho_0}\right)^{\frac{1}{2}}\left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{n-1}{2n}}, \qquad h_0 - h_{1*} = \left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{n-1}{2n}}\left(\frac{\rho_0}{-\sigma^\circ}\right)^{\frac{1}{2}} t$$

$$\varepsilon_{1*} = -\left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{1}{n}}, \qquad \frac{\rho_1}{\rho_0} = \left[1 - \left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{1}{n}}\right]^{-1}, \ \sigma_1(h, t) = \sigma_0$$

For the reflected wave

$$h_{2*}' = -q h_1{}'_*, \qquad h_{2*} = -q h_{1*}$$

$$\rho_2'(h) = \rho_0\left[1 - \frac{q+1}{q}\left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{1}{n}}\right]^{-1}, \qquad \sigma_2(t) = (1+q)\,\sigma_0$$

where q is the positive root of equation (3.14).

b) Suppose that the given stress suddenly assumes a constant value σ_0

and after a finite interval of time (after the front has travelled a finite distance l from the initial section) suddenly becomes zero once more, so that

$$\sigma_0(h_{1*}) = \begin{cases} 0 & (h_{1*} > h_0) \\ \sigma_0 & (h_0 - l < h_{1*} < h_0) \\ 0 & (h_{1*} < h_0 - l) \end{cases}$$

The incident wave can then be defined by the following functions

$$-h_{1*}' = \left(\frac{-\sigma^\circ}{\rho_0}\right)^{\frac{1}{2}} \left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{n-1}{2n}} \left(\frac{l}{h_0 - h_{1*}}\right)^{\frac{n-1}{n+1}}, \qquad \varepsilon_1(h_{1*}) = \left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{1}{n}} \left(\frac{l}{h_0 - h_{1*}}\right)^{\frac{2}{n+1}}$$

$$t = \left(\frac{\rho_0}{-\sigma^\circ}\right)^{\frac{1}{2}} \left(\frac{\sigma^\circ}{\sigma_0}\right)^{\frac{n-1}{2n}} l \left\{1 + \frac{n+1}{2n}\left[\left(\frac{h_0 - h_{1*}}{l}\right)^{\frac{2n}{n+1}} - 1\right]\right\}$$

$$\frac{\rho_1(h_{1*})}{\rho_0} = \left[1 - \left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{1}{n}} \left(\frac{l}{h_0 - h_{1*}}\right)^{\frac{2}{n+1}}\right]^{-1}, \qquad \sigma_1(h, t) = \sigma_0 \frac{h - h_0}{h_{1*} - h_0}\left(\frac{l}{h_0 - h_{1*}}\right)^{\frac{2n}{n+1}}$$

The last five formulas apply for $h_{1*} < h_0 - l$. For $h_0 - l < h_{1*} < h_0$ the stress remains constant and the wave generated is the same as the wave in example (a) (it has constant parameters). The formulas given above show that the front velocity, the deformation, the density, and the stress on the front decrease monotonically with the propagation.

Let us now consider the reflected wave.

From (3.12) we obtain

$$h_{2*}' = -\sqrt{n}\left(\frac{h_0 - h_{1*}}{h_0 - h_{2*}}\right)^{\frac{n-1}{n+1}} h_{1*}' = H\left(\frac{l}{h_0 - h_{2*}}\right)^{\theta}$$

$$H = \sqrt{n}\left(\frac{-\sigma^\circ}{\rho_0}\right)^{\frac{1}{2}}\left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{n-1}{2n}}, \qquad \theta = \frac{n-1}{n+1} \tag{4.1}$$

It follows from (4.1) that the velocity of the front of the reflected wave increases with increase in its distance from the obstacle. This is explained by the fact that as it propagates, the reflected wave passes through a medium the density of which is constantly increasing.

The velocity of the reflected wave-front varies from a value $H(l/h_0)^\theta$ at the obstacle to a value H at $h_{2*} = h_0 - l$.

The density $\rho_2(h)$ also increases with h. In fact

$$\frac{\rho_2(h)}{\rho_0} = \frac{\rho_1}{\rho_0 + \rho_1 v_1 / h'_{2*}}$$

Finally, the magnitude of the step in the stress on the wave-front is given by

$$\sigma_2 - \sigma_b = \frac{\sigma_0 \sqrt{n}}{h_0 - h_{1*}} \, l^{\frac{2n}{n+1}} (h_0 - h_{2*})^{-\frac{n-1}{n+1}} \tag{4.2}$$

5. Reflection of a plane wave from a massive obstacle. In accordance with the conditions of a plane wave, we shall assume that the obstacle is bounded by two parallel planes. Without restricting the generality, we shall assume that the obstacle is an infinitely thin plate possessing an infinite surface density.

Suppose that the wave strikes the plate normally. As a result of the collision with the incident wave the plate is set in motion, and on the other side of the plate a new wave motion is generated - a transmitted wave.

Let us restrict the problem and study the phenomenon for a time interval which is sufficiently small for us to be able to make the following assumptions:

1) the stress is higher in the reflected wave than in the incident wave (there exists a shock-front in the reflected wave);

2) the plate moves in one direction (in the direction of propagation of the wave);

3) as a result of its motion the plate generates a compression wave which produces a monotonically increasing stress on the boundary plane.

These assumptions are hypotheses the validity of which can be checked in an actual derivation of a solution.

The intensity of the wave generated behind the obstacle will depend on the parameters of the incident wave and the mass of the obstacle.

Let us first consider the case when the transmitted wave is weak and can be described by the linear theory. This will be the case for a sufficiently massive obstacle.

For the region of the transmitted wave, which we will denote by the suffix 4, we can write down the following relations:

$$\sigma_4 = k\varepsilon_4, \qquad k = \lambda + 2\mu \qquad (\lambda,\ \mu \text{ —are Lamé parameters})$$

$$\frac{\partial \sigma_4}{\partial h} - \rho_0 \frac{\partial v_4}{\partial t} = 0, \qquad v_4 = \frac{\partial u_4}{\partial t}, \qquad \varepsilon = \frac{\partial u_4}{\partial h} \tag{5.1}$$

Consequently, the displacement u_4 satisfies the equation

$$\frac{\partial^2 u_4}{\partial h^2} - \frac{1}{a^2}\frac{\partial^2 u_4}{\partial t^2} = 0, \qquad a^2 = \frac{k}{\rho_0}$$

For the transmitted wave propagating in the direction of the negative axis, we have

$$u_4(h, t) = f(h + at)$$

It is required to define the reflected and transmitted waves and the motion of the obstacle.

In order to determine the required functions we have the following conditions:

on the reflected shock-wave

$$(D - v_2)\rho_2 = (D - v_1)\rho_1, \qquad \sigma_b - \sigma_2 = (D - v_2)\rho_2(v_2 - v_1)$$

on the obstacle ($h = 0$)

$$x_4(0, t) = x_{20}(0, t) = U(t), \qquad v_2(h, t) = x_{20}'(t) = U'(t)$$

The motion of the obstacle is governed by the equation

$$\sigma_2(0, t) = \sigma_4(0, t) = MU''(t)$$

Here M, U are the mass and displacement of the obstacle. As before, for the reflected wave we have

$$x_2(h, t) = \int_0^h \frac{\rho_0\, d\eta}{\rho_2(\eta)} + x_{20}(t), \qquad \sigma_2(h, t) = \rho_0 x_{20}''(t)\, h + \sigma_2(0, t)$$

The velocity D of the reflected shock-wave is given by the relation

$$D = \frac{dx_{2*}}{dt} = \frac{\rho_0}{\rho_2(h_{2*})} h_{2*}' + v_2$$

These nine equations enable us to determine the functions

$$U(t),\ \ \sigma_2(0, t),\ \ \rho_2(h),\ \ h_{2*}(t),\ \ x_4(h + at),\ \ v_2(t),\ \ x_{20}(t),\ \ D(t),\ \ \sigma_2(h, t)$$

The conditions on the shock-wave can be written in the form

$$v_1 - v_2 = \left(\frac{\rho_0}{\rho_2} - \frac{\rho_0}{\rho_1}\right) h_{2*}', \qquad \sigma_b - \sigma_2 = -\rho_0 h_{2*}'(v_2 - v_1) \tag{5.2}$$

Since $\rho_0/\rho_2 - \rho_0/\rho_1 = \varepsilon_2 - \varepsilon_1$, and on the shock-front $\sigma_2 = \sigma^\circ|\varepsilon_2|^n$, we obtain from (5.2)

$$\sigma^\circ\left[-\varepsilon_1(h_{2*}) - \frac{v_1 - v_2}{h_{2*}'}\right]^n + \sigma_b = \rho_0 h_{2*}'(v_2 - v_1)$$

For the particle velocity in the incident wave we have

$$v_1 = -\varepsilon_1(h_{1*})\,h_{1*}' = -\frac{\sigma_1(h_{1*})}{\rho_0 h_{1*}'}$$

Therefore

$$\sigma^\circ[-\varepsilon_1(h_{2*})]^n\left[1-\frac{\varepsilon_1(h_{1*})\,h_{1*}'+v_2}{\varepsilon_1(h_{2*})\,h_{2*}'}\right]^n+\sigma_b=\rho_0 h_{2*}'\left(v_2+\frac{\sigma_{1*}}{\rho_0 h_{1*}'}\right)$$

or, bearing in mind that

$$\sigma^\circ[-\varepsilon_1(h_{2*})]^n=\sigma_b$$

$$\left[1-\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\frac{h_{1*}'}{h_{2*}'}-\frac{v_2}{\varepsilon_1(h_{2*})h_{2*}'}\right]^n=1-\frac{\sigma_{1*}}{\sigma_b}\frac{h_{2*}'}{h_{1*}'}-\frac{\rho_0 h_{2*}' v_2}{\sigma_b}$$

Expressing the stress ratio σ_{1*}/σ_b in terms of the ratio of the corresponding strains and rearranging, we obtain finally

$$\left[1-\beta\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\frac{h_{1*}'}{h_{2*}'}\right]^n=1-\left[\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\right]^n\beta\frac{h_{2*}'}{h_{1*}'}\qquad\left(\beta=1+\frac{v_2}{\varepsilon_1(h_{1*})h_{1*}'}=1-\frac{v_2}{v_1}\right)\tag{5.3}$$

This is a nonlinear first-order differential equation in the unknown function h_{2*}'; it is similar to equation (3.9) and differs only in the multiplier β, which depends on the unknown particle velocity in the reflected wave. Thus equation (5.3) contains two unknown functions, and in order to solve the problem a further equation is needed. This will be provided by the so far unused equation of motion of the obstacle.

In order to determine the function $\sigma_2(0,\ t)$ we make use of (5.3), bearing in mind that $U' = v_2$. Then

$$\sigma_2(0,t)=\sigma_b-\rho_0 h_{2*}'(v_2-v_1)-\rho_0 h_{2*}v_2'$$

From (5.1) the stress $\sigma_4(h,\ t)$ is given by

$$\sigma_4(h,t)=kf'(h+at)$$

At the obstacle this stress is

$$\sigma_4(0,t)=kf'(at)=\frac{k}{a}v_2$$

As a result the equation of motion of the obstacle in expanded form becomes the following linear first-order differential equation in v_2:

$$\frac{dv_2}{dt}+P(t)\,v_2=Q(t)\qquad\left(P(t)=\frac{k/a+\rho_0 h_{2*}'}{M+\rho_0 h_{2*}},\ \ Q(t)=\frac{\sigma_b+\rho_0 h_{2*}'v_1}{M+\rho_0 h_{2*}}\right)$$

The solution to this equation, which vanishes at the instant of collision of the incident wave with the obstacle (at $t = t_0$), is

$$v_2(t) = \frac{1}{M + \rho_0 h_{2*}} \int_{t_0}^{t} (\sigma_b + \rho_0 h_{2*}' v_1) \exp\left(-\frac{k}{aM}\int_{\tau}^{t} \frac{d\tau'}{1 + \rho_0 h_{2*}/M}\right) d\tau$$

This formula can be easily rewritten as follows:

$$v_2(t) = \frac{\sigma_b}{M + \rho_0 h_{2*}} \int_{t_0}^{t} \left[1 - \left[\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\right]^n \frac{h_{2*}'}{h_{1*}'}\right] \exp\left(-\frac{k}{aM}\int_{\tau}^{t} \frac{d\tau'}{1 + \rho_0 h_{2*}/M}\right) d\tau \qquad (5.4)$$

Equations (5.3) and (5.4) together determine both the unknown functions h_{2*} and v_2. Elimination of v_2 leads to a single integro-differential equation in h_{2*}/h_{1*}. When h_{2*} and v_2 are known the remaining unknown functions can be easily expressed in terms of them.

For instance, the pressure of the reflected wave on the obstacle is given by

$$\sigma_2(0, t) = \frac{\rho_0 \sigma_b}{M} \frac{k h_{2*}/aM - h_{2*}'}{(1 + \rho_0 h_{2*}/M)^2} \int_{t_0}^{t} \left(1 - \left|\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\right|^n \frac{h_{2*}'}{h_{1*}'}\right) \exp\left(-\frac{k}{aM}\int_{\tau}^{t} \frac{d\tau'}{1 + \rho_0 h_{2*}/M}\right) d\tau +$$
$$+ \sigma_b \left(1 - \left|\frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})}\right|^n \frac{h_{2*}'}{h_{1*}'}\right) \left(1 + \frac{\rho_0 h_{2*}}{M}\right)^{-1} \qquad (5.5)$$

The values of the required functions at the instant of reflection (at $t = t_0$) can be found simply. At $t = t_0$ we have $h_{1*} = h_{2*} = 0$, $v_2 = 0$, $\beta = 1$. Therefore, from (5.3) we obtain $h_{2*}'/h_{1*}' = q$, where q is the root of equation (3.14), and $q > 1$.

For the coefficient of reflection we obtain from (5.5), by setting $t = t_0$, the expression

$$\sigma_2(0, t)/\sigma_b = 1 + q$$

This is the same result as for a fixed obstacle. The explanation lies in the fact that at the instant $t = t_0$ the obstacle is stationary. The stress $\sigma_4(0, t)$ on the rear face of the obstacle varies in the same way as the velocity of the obstacle, which follows from the formula

$$\sigma_4(0, t) = k v_2 / a$$

Formulas (5.3) to (5.5) show that the hypotheses made concerning the nature of the phenomenon are justified for a certain finite interval of time after the instant of reflection. In order to solve equations (5.3) and (5.4) in their exact form we have to resort to numerical methods.

However, as in Section 4, we can adopt the method of replacing equation (5.3) by a simpler approximate expression. From physical considerations it is clear that $0 \leqslant \beta \leqslant 1$ and

$$0 < \frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})} \leqslant 1, \qquad \frac{h_{1*}'}{h_{2*}'} < 1$$

Assuming, therefore, that

$$\left| \beta \frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})} \frac{h_{2*}'}{h_{1*}'} \right| < 1$$

we expand the left-hand side of equation (5.3) into a power series and retain two terms only. As a result we obtain

$$\frac{h_{2*}'}{h_{1*}'} = -\sqrt{n} \left| \frac{\varepsilon_1(h_{1*})}{\varepsilon_1(h_{2*})} \right|^{\frac{n-1}{2}} + \ldots$$

In this approximation we see that the same law results in this case as in the case of a fixed obstacle.

Finally, let us study in more detail the particular case when the incident wave is in the form of a step (Section 4, example (a)). The basic formulas are then very much simplified, and if one further approximation is made, the problem can be solved in an explicit and simple form. We are then able as a particular typical example to investigate the influence of a massive obstacle on the readings of strain- or velocity-gauges attached to the obstacle. In this way the results obtained can be applied to the theory of instruments used for measuring stress and particle velocity in the propagation of waves in soils. The present formulation of the problem for plane waves does not, of course, take account of the phenomenon whereby a wave "bypasses" an obstacle, which occurs in actual practice.

Suppose an incident wave in the form of a step strikes an obstacle; then

$$\sigma_1(h, t) = \sigma_0, \qquad \sigma_b = \sigma_0$$

$$h_{1*}' = -\sqrt{\frac{-\sigma^\circ}{\rho_0}} \left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{n-1}{2n}}, \qquad \varepsilon_{1*}(h) = -\left(\frac{\sigma_0}{\sigma^\circ}\right)^{\frac{1}{n}}$$

Since $\varepsilon_1(h_{1*})/\varepsilon_1(h_{2*}) = 1$, eqation (5.3) can be written in the simplified form

$$\left(1 - \beta \frac{h_{1*}'}{h_{2*}'}\right)^n = 1 - \beta \frac{h_{2*}'}{h_{1*}'} \qquad \left(\beta = 1 - \frac{v_2}{v_1}, \quad v_1 = \text{const}\right)$$

The relation between h_{2*}'/h_{1*}' and β can be expressed in parametric form, which is convenient for the graphical solution of the equation

$$\frac{h'_{2*}}{h'_{1*}} = \left(\frac{z^n - 1}{z - 1}\right)^{\frac{1}{2}}$$

$$\beta = \sqrt{(z^n - 1)(z - 1)}$$

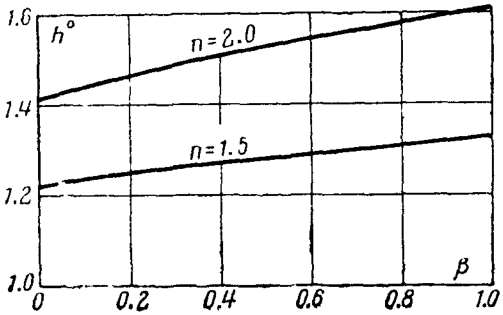

Fig. 7.

In Fig. 7 this relation is shown graphically for $n = 1.5$ and $n = 2.0$. We see that for $0 \leqslant \beta \leqslant 1$ the ratio h'_{2*}/h'_{1*} varies only very slightly (within approximately 10%). This ratio can therefore be taken as approximately constant and equal to q (to its value at $t = t_0$). This assumption enables us to obtain an expression for the stress $\sigma_2(0, t)$ and the velocity v_2 in a final simple form. We introduce the nondimensional quantities

$$\tau = \frac{(t - t_0)k}{aM}, \qquad V_2 = \frac{v_2}{a}, \qquad V_1 = \frac{v_1}{a}, \qquad -C = \frac{h'_{1*}\rho_0 a}{k} = \frac{h'_{1*}}{a}$$

Formula (5.4) then yields

$$V_2 = +\frac{\sigma_0}{k}\frac{1+q}{1+qC}\left[1 - (1 + qC\tau)^{-\left(1+\frac{1}{qC}\right)}\right] \tag{5.6}$$

For the velocity V_1 in the incident wave we obtain

$$V_1 = \frac{\sigma_0}{k}\frac{1}{C}$$

The obstacle is set into motion with a speed monotonically increasing from zero and tending with time to the limiting value

$$\frac{\sigma_0}{k}\frac{1+q}{1+qC}$$

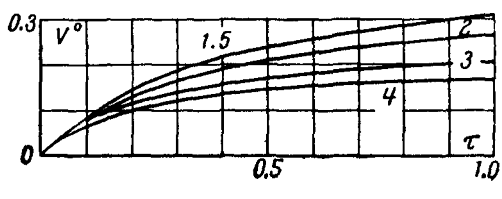

Fig. 8.

For $C > 1$ this limiting value is greater than v_1, and for $C < 1$ it is smaller. Formula (5.6) is interpreted graphically in Fig. 8 for values of $qC = 1.5$, 2, 3, 4, where $V^o = kV_2/\sigma_0(1 + q)$.

From (5.5) the pressure of the incident wave on the obstacle is given by

$$\frac{\sigma_2(0, t)}{\sigma_0} = \frac{1+q}{1+qC\tau}\left\{1 - \frac{qC(1-\tau)}{1+qC}\left[1 - (1 + qC\tau)^{-\left(1+\frac{1}{qC}\right)}\right]\right\} \tag{5.7}$$

Hence we see that at the instant of collision of the incident wave with the obstacle the stress instantaneously attains the value $\sigma_0(1+q)$, as in the case of a fixed wall, and then decays. Then

$$\sigma_2(0, t) \to \sigma_0 \frac{1+q}{1+qC} \quad \text{as} \quad \tau \to \infty$$

This limiting value is less than σ_0 if $C > 1$; if $C < 1$ the limiting value is greater than σ_0. The results of calculations based on formula

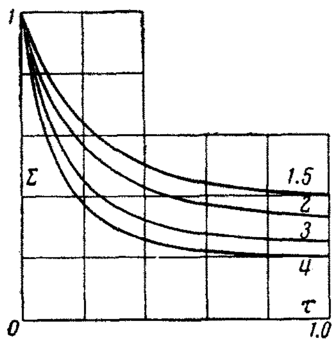

Fig. 9.

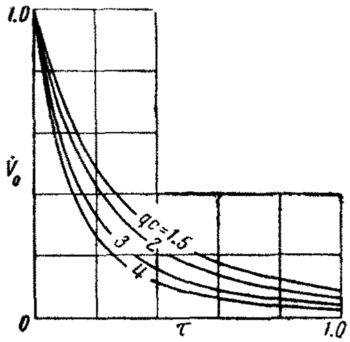

Fig. 10.

(5.7) for values of $qC = 1.5,\ 2,\ 3,\ 4$ are shown in Fig. 9, where

$$\Sigma = \sigma_2(0, t) / \sigma_0(1+q)$$

Making use of formula (5.7) and the relation

$$\sigma_2(h, t) = \sigma_2(0, t) + \rho_0 \frac{dv_2}{dt} h$$

we can determine the stress at the front of the reflected wave:

$$\frac{\sigma(h_{2*}, t)}{\sigma_0} = \frac{1+q}{1+qC}\left[1 + (1+qC\tau)^{-\left(1+\frac{1}{qC}\right)}\right] \tag{5.8}$$

By differentiating equation (5.6) we obtain the acceleration which the obstacle experiences during its motion. For the nondimensional derivative we obtain

$$\frac{dV_2}{d\tau} = -\frac{\sigma_0(1+q)}{k}[1+qC\tau]^{-\left(2+\frac{1}{qC}\right)}$$

The results of calculations based on this formula are given in Fig. 10, where

$$\dot{V}^\circ = M\dot{v}_2 / \sigma_0(1+q)$$

This derivative is connected to the true acceleration by the relation

$$\frac{dv_2}{dt} = \frac{k}{M}\frac{dV_2^\circ}{d\tau}$$

The maximum acceleration occurs at the instant of collision of the incident wave with the obstacle, when it has the (absolute) value

$$W = \frac{\sigma_0 (1 + q)}{M}$$

It is expedient to express this acceleration in multiples of the acceleration g due to gravity:

$$\frac{W}{g} = \frac{\sigma_0}{Q}(1 + q)$$

where Q is the weight of the obstacle per unit area.

It should be remembered that formulas (5.6), (5.7) and (5.8) were derived on the assumption that the medium deforms linearly (during loading), that the stress on the front of the reflected wave is greater than that on the incident wave, and that in the transmitted wave unloading has not commenced. In particular, the condition $\sigma_2(h_{2*}, t) > \sigma_0$ can be written with the aid of formula (5.8), which requires that

$$\tau < \tau_0 = \frac{1}{qC}\left[\left(\frac{C(1+q)}{C-1}\right)^{\frac{qC}{1+qC}} - 1\right]$$

The value of τ_0 corresponds to the instant at which the reflected shock wave disappears. Note that at this instant the velocity v_2 of the obstacle becomes equal to the particle velocity v_1 in the incident wave.

The motion when $C < 1$ cannot be investigated in the present study, since it was assumed that the incident wave travels through an undisturbed medium. If $C < 1$, an elastic wave must precede the plastic wave. This case requires a special investigation. Other conditions governing the applicability of the theory must be considered for each specific problem.

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